1. The digital sum of 13950264876 is 51. Since $51 = 17 \times 3$, it is a multiple of three, but it is not a multiple of nine. Thus the prime factorisation of 13950264876 contains 3, but not $3^2$. Thus it cannot be a square.

2. Let the integer we are looking for be $n$ and the common remainder be $r$. Then there are integers $a$, $b$ and $c$ such that $364 = an + r$, $414 = bn + r$ and $539 = cn + r$. But then $414 − 364 = (b − a)n$, $539 − 414 = (c − b)n$ and $539 − 364 = (c − a)n$, so we can see that $n$ divides the differences of the three numbers. Now

\[
\begin{align*}
414 - 364 &= 50 \\
539 - 414 &= 125 \\
539 - 364 &= 175
\end{align*}
\]

The largest number that divides all three differences is 25.

3. We will first show that $\triangle AEF$ is similar to $\triangle ACB$, with a scale factor of $\frac{1}{2}$.

Since $E$ is the midpoint of $AC$, $AE = \frac{1}{2}AC$. Similarly, $AF = \frac{1}{2}AB$. Furthermore, $\angle A$ is common to both triangles. So $\triangle AEF \sim \triangle ACB$, (ASS), with a scale factor of $\frac{1}{2}$.

As a result, $EF = \frac{1}{2}BC$, and $\angle AEF = \angle ACB$. Thus $EF \parallel BC$. 

\[
\begin{align*}
A & \quad E \quad F \\
C & \quad B
\end{align*}
\]
4. Since acute angled triangles exist, we know that there are convex polygons with at least three acute angles.

If the interior angle of a polygon is acute, then the exterior angle must be obtuse. The sum of the exterior angles of a convex polygon is $360^\circ$. As the sum of four numbers greater than 90 is greater than 360, a convex polygon must have less than four acute angles. Thus three is the largest number of acute angles that a convex polygon can have.

5. Let $a$ and $b$ be integers such that $a^3 = x + \sqrt{x^2 + 1}$ and $b^3 = x - \sqrt{x^2 + 1}$. Then

$$\sqrt[3]{x + \sqrt{x^2 + 1}} + \sqrt[3]{x - \sqrt{x^2 + 1}} = a + b = y,$$

where $y \in \mathbb{Z}$. Now

$$(a + b)^3 = a^3 + b^3 + 3ab(a + b),$$

but

$$a^3 + b^3 = x + \sqrt{x^2 + 1} + x - \sqrt{x^2 + 1} = 2x$$

$$ab = \sqrt[3]{(x + \sqrt{x^2 + 1})(x - \sqrt{x^2 + 1})}$$

$$= \sqrt[3]{x^2 - (x^2 + 1)}$$

$$= \sqrt[3]{-1} = -1.$$

Consequently,

$$y^3 = 2x - 3y$$

$$\therefore x = \frac{y^3 + 3y}{2}, \quad y \in \mathbb{Z}.$$

We can see that $y^3$ and $3y$ have the same parity, and so $x$ is an integer.

6. (a) $\phi(12) = 4$ and $\phi(30) = 8$.

(b) We can think of $\phi(n)$ as being the number of positive integers less than $n$ which are not a multiple of a factor of $n$ (except the factor 1). So if $p$ is prime, its only factors are 1 and $p$. Thus $\phi(p) = p - 1$.

For $p^2$, the factors are 1, $p$ and $p^2$, so the multiples of the factors that aren’t 1 are $p,2p,3p,\ldots,p^2$, of which there are $p$. So $\phi(p^2) = p^2 - p = p(p - 1)$.

For $p^3$, the factors are $1, p, p^2$ and $p^3$. Multiples of the factors that aren’t 1 are $p,2p,3p,\ldots,p^2,(p + 1)p,\ldots,2p^2,\ldots,p^3$. That is, a total of $p^2$ factors. So $\phi(p^3) = p^3 - p^2 = p^2(p - 1)$.

(c) Using the same method as above, the factors of $pq$ are 1, $p, q$ and $pq$. The multiples of the factors that aren’t 1 are $p,2p,\ldots,qp$ (q multiples) and $q,2q,\ldots,pq$ (p multiples), but we don’t want to count $pq$ twice. So $\phi(pq) = pq - q - p + 1 = (p - 1)(q - 1)$. 


Senior Questions

1. (a) Expanding the right hand side,

\[(n^2 - 3n - 1)^2 - 25n^2 = n^4 - 2n^2(3n + 1) + (3n + 1)^2 - 25n^2\]
\[= n^4 - 6n^3 - 2n^2 + 9n^2 + 6n + 1 - 25n^2\]
\[= n^4 - 6n^3 - 18n^2 + 6n + 1\]

(b) We can use the previous result to factorise \(n^4 - 6n^3 - 18n^2 + 6n + 1\). So

\[(n^2 - 3n - 1)^2 - 25n^2 = [(n^2 - 3n - 1) - 5n][(n^2 - 3n - 1) + 5n]\]
\[= (n^2 - 8n - 1)(n^2 + 2n - 1)\]

Now if \(n^4 - 6n^3 - 18n^2 + 6n + 1\) is prime, then one or the other of these factors must be one. But then either \(n^2 - 8n - 2 = 0\) or \(n^2 + 2n - 2 = 0\), and in both cases, \(\Delta\) is not a square number, so there are no integer solutions.

2. From the graph below, we can see there are 7 real roots.

So

\[f(x) = x - 3\pi(1 - \sin x)\]
\[f'(x) = 3\pi \cos x + 1\]

We should probably use \(x_0 = \frac{\pi}{4}\) as a starting value. Then

\[x_{n+1} = x_n - \frac{x_n - 3\pi(1 - \sin x_n)}{3\pi \cos x_n + 1}\]
\[x_1 = 1.04309281268308\]
\[x_2 = 1.08468800536226\]
\[x_3 = 1.08600204983348\]
\[x_4 = 1.0860038435583\]
\[x_5 = 1.0860038435721\]
Using the symmetry of the graph, we can estimate that the largest root is
\[
x_{\text{max}} = \frac{11\pi}{2} + \left(\frac{\pi}{2} - x_5\right) \approx 17.763552537181550
\]
\[
f(x_{\text{max}}) = 3.55 \times 10^{-15}
\]

3. Let \(ABC\) be a triangle, and let \(D, E\) and \(H\) be the midpoints of \(BC, AC\) and \(AB\), respectively. Suppose that \(O\) is the point of intersection of \(BE\) and \(AD\). Let \(F\) and \(G\) be the midpoints of \(OA\) and \(OB\), respectively. Then, applying the mid-line theorem to \(\triangle AOB\), \(FG\parallel AB\), and \(FG = \frac{1}{2}AB\). Similarly, by applying the mid-line theorem to \(\triangle ACB\), we can see that \(ED = \frac{1}{2}AB\) and \(ED\parallel AB\). Thus \(DEFG\) is a parallelogram, and \(O\) is the point of intersection of its two diagonals. Thus \(OD = OF = AF\) and \(OE = OG = GB\). Consequently, \(O\) is located \(\frac{1}{3}\) the way along the medians \(AD\) and \(BE\) from their respective feet.

By a similar argument, we can show that the point of intersection of the medians \(CH\) and \(AD\) lies \(\frac{1}{3}\) the length of \(AD\) away from \(D\). Thus the two points of intersection coincide, and the three medians are concurrent.