MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 8, July 2, 2018

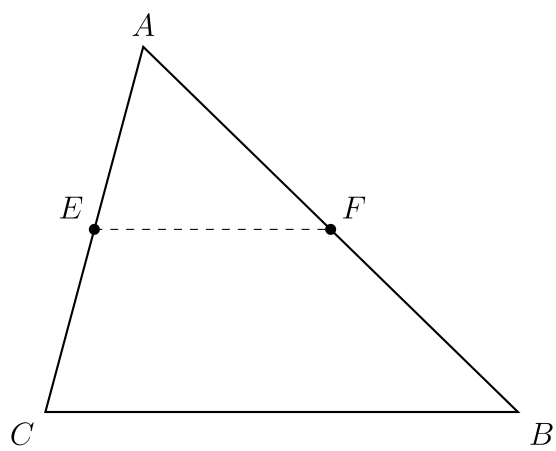
1. The digital sum of 13950264876 is 51. Since $51 = 17 \times 3$, it is a multiple of three, but it is not a multiple of nine. Thus the prime factorisation of 13950264876 contains 3, but not 3^2. Thus it cannot be a square.

2. Let the integer we are looking for be n and the common remainder be r. Then there are integers a, b and c such that $364 = an + r$, $414 = bn + r$ and $539 = cn + r$. But then $414 - 364 = (b - a)n$, $539 - 414 = (c - b)n$ and $539 - 364 = (c - a)n$, so we can see that n divides the differences of the three numbers. Now

$$\begin{aligned} 414 - 364 &= 50 \\ 539 - 414 &= 125 \\ 539 - 364 &= 175 \end{aligned}$$

   The largest number that divides all three differences is 25.

3. We will first show that $\triangle AEF$ is similar to $\triangle ACB$, with a scale factor of $\frac{1}{2}$.



   Since E is the midpoint of AC, $AE = \frac{1}{2}AC$. Similarly, $AF = \frac{1}{2}AB$. Furthermore, $\angle A$ is common to both triangles. So $\triangle AEF \sim \triangle ACB$, (ASS), with a scale factor of $\frac{1}{2}$.

   As a result, $EF = \frac{1}{2}BC$, and $\angle AEF = \angle ACB$. Thus $EF \| BC$.

4. Since acute angled triangles exist, we know that there are convex polygons with at least three acute angles.

   If the interior angle of a polygon is acute, then the exterior angle must be obtuse. The sum of the exterior angles of a convex polygon is 360°. As the sum of four numbers greater than 90 is greater than 360, a convex polygon must have less than four acute angles. Thus three is the largest number of acute angles that a convex polygon can have.

5. Let a and b be integers such that $a^3 = x + \sqrt{x^2+1}$ and $b^3 = x - \sqrt{x^2+1}$. Then $\sqrt[2]{x+\sqrt{x^2+1}} + \sqrt[3]{x-\sqrt{x^2+1}} = a + b = y$, where $y \in \mathbb{Z}$. Now

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b),$$

   but

$$\begin{aligned} a^3 + b^3 &= x + \sqrt{x^2+1} + x - \sqrt{x^2+1} = 2x \\ ab &= \sqrt[3]{(x+\sqrt{x^2+1})(x-\sqrt{x^2+1})} \\ &= \sqrt[3]{x^2-(x^2+1)} \\ &= \sqrt[3]{-1} = -1. \end{aligned}$$

   Consequently,

$$\begin{aligned} y^3 &= 2x - 3y \\ \therefore x &= \frac{y^3+3y}{2}, \qquad y \in \mathbb{Z}. \end{aligned}$$

   We can see that y^3 and $3y$ have the same parity, and so x is an integer.

6. (a) $\phi(12) = 4$ and $\phi(30) = 8$.

   (b) We can think of $\phi(n)$ as being the number of positive integers less than n which are not a multiple of a factor of n (except the factor 1). So if p is prime, its only factors are 1 and p. Thus $\phi(p) = p - 1$.

   For p^2, the factors are 1, p and p^2, so the multiples of the factors that aren't 1 are $p, 2p, 3p, \ldots, p^2$, of which there are p. So $\phi(p^2) = p^2 - p = p(p-1)$.

   For p^3, the factors are 1, p, p^2 and p^3. Multiples of the factors that aren't 1 are $p, 2p, 3p, \ldots, p^2$, $(p+1)p, \ldots, 2p^2, \ldots, p^3$. That is, a total of p^2 factors. So $\phi(p^3) = p^3 - p^2 = p^2(p-1)$.

   (c) Using the same method as above, the factors of pq are 1, p q and pq. The multiples of the factors that aren't 1 are $p, 2p, \ldots, qp$ (q multiples) and $q, 2q, \ldots, pq$ (p multiples), but we don't want to count pq twice. So $\phi(pq) = pq - q - p + 1 = (p-1)(q-1)$.

Senior Questions

1. (a) Expanding the right hand side,

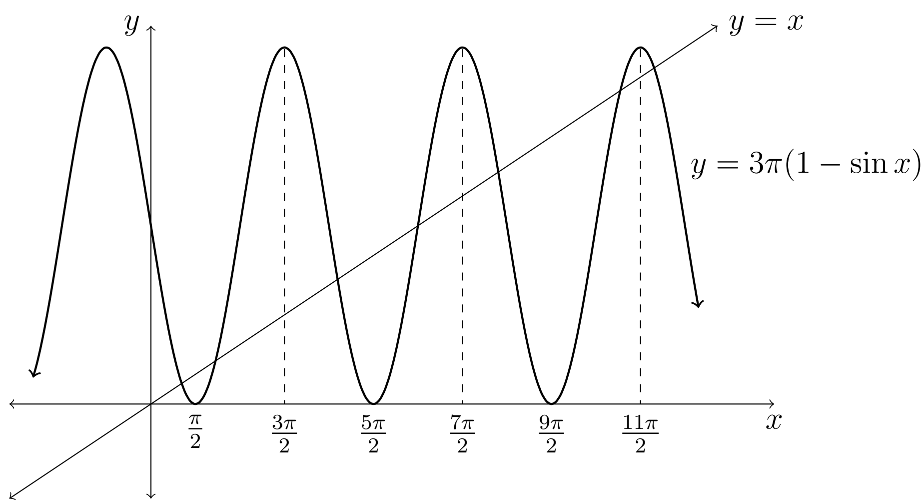
$$\begin{aligned}(n^2-3n-1)^2-25n^2 &= n^4-2n^2(3n+1)+(3n+1)^2-25n^2\\ &= n^4-6n^3-2n^2+9n^2+6n+1-25n^2\\ &= n^4-6n^3-18n^2+6n+1\end{aligned}$$

   (b) We can use the previous result to factorise $n^4-6n^3-18n^2+6n+1$. So

$$\begin{aligned}(n^2-3n-1)^2-25n^2 &= [(n^2-3n-1)-5n][(n^2-3n-1)+5n]\\ &= (n^2-8n-1)(n^2+2n-1)\end{aligned}$$

   Now if $n^4-6n^3-18n^2+6n+1$ is prime, then one or the other of these factors must be one. But then either $n^2-8n-2=0$ or $n^2+2n-2=0$, and in both cases, Δ is not a square number, so there are no integer solutions.

2. From the graph below, we can see there are 7 real roots.



So

$$\begin{aligned}f(x) &= x-3\pi(1-\sin x)\\ f'(x) &= 3\pi\cos x+1\end{aligned}$$

We should probably use $x_0=\frac{\pi}{4}$ as a starting value. Then

$$\begin{aligned}x_{n+1} &= x_n-\frac{x_n-3\pi(1-\sin x_n)}{3\pi\cos x_n+1}\\ x_1 &= 1.04309281268308\\ x_2 &= 1.08468800536226\\ x_3 &= 1.08600204983348\\ x_4 &= 1.08600338435583\\ x_5 &= 1.08600338435721\end{aligned}$$

Using the symmetry of the graph, we can estimate that the largest root is

$$
\begin{aligned}
x_{max} &= \frac{11\pi}{2} + \left(\frac{\pi}{2} - x_5\right) \approx 17.763552537181550 \\
f(x_{max}) &= 3.55 \times 10^{-15}
\end{aligned}
$$

3. Let ABC be a triangle, and let D, E and H be the midpoints of BC, AC and AB, respectively. Suppose that O is the point of intersection of BE and AD. Let F and G be the midpoints of OA and OB, respectively. Then, applying the mid-line theorem to $\triangle AOB$, $FG \| AB$, and $FG = \frac{1}{2}AB$. Similarly, by applying the mid-line theorem to $\triangle ACB$, we can see that $ED = \frac{1}{2}AB$ and $ED \| AB$. Thus $DEFG$ is a parallelogram, and O is the point of intersection of its two diagonals. Thus $OD = OF = AF$ and $OE = OG = GB$. Consequently, O is located $\frac{1}{3}$ the way along the medians AD and BE from their respective feet.

   By a similar argument, we can show that the point of intersection of the medians CH and AD lies $\frac{1}{3}$ the length of AD away from D. Thus the two points of intersection coincide, and the three medians are concurrent.